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A Weierstrass Theorem for Real, Separable Hilbert Spaces*

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Let K be a compact subset of a real, separable Hilbert space H and let $C(K)$ denote the family of continuous functions (operators) from H into H together with the uniform norm topology

$$\| f - g \| = \max_{x \in K} \| f(x) - f(x) \|.$$

We prove that the Weierstrass Theorem holds on $C(K)$. That is, the continuous polynomial operators on H are dense in $C(K)$. A polynomial operator

$$Px = L_0 + L_1 x + L_2 x^2 + \cdots + L^n x_n$$

of degree n is defined by means of the k-linear operators L_k ; $L_k x^k$ denotes L_k applied to the k-tuple $(x, x,..., x)$.

1. INTRODUCTION

Let X and Y be normed linear spaces and let K be a compact subset of X. Let $C(K)$ denote the space of continuous functions from X to Y restricted to K, where $C(K)$ carries the uniform norm topology

$$\| f - g \| = \max_{x \in K} \| f(x) - g(x) \|.$$

In the event $X = Y$ is the real line, the classical Weierstrass Theorem states that the family of polynomials on X is dense in $C(K)$. If $X = E^n$, $Y = E$, where E is the real line, a straightforward application of the Stone–Weierstrass Theorem proves that the polynomials in n real variables, $n = 1, 2,...,$ are dense in $C(K)$. In this paper we prove a Hilbert space analog to the Weierstrass Theorem. That is, we prove (Theorem 5.5) that if $X = Y = H$ is a real, separable Hilbert space, then the family of all continuous polynomials

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from H to H is dense in $C(K)$. This necessitates the definition of a polynomial (polynomial operator) in a linear space.

2. Polynomials in a Linear Space

Let X be a linear space over the field of real (complex) numbers. Let $k \geqslant 1$ and let X^k denote the direct product

$$\underbrace{X \times X \times \cdots \times X}_{k \text{ times}}$$

A *k-linear operator M, on X*, is a function on X^k into X which is linear in each of its arguments separately. That is, for each $i = 1, 2,..., k$,

$$\begin{aligned} M(x_1, x_2, ..., x_i + y_i, ..., x_k) \\ = M(x_1, ..., x_i, ..., x_k) + M(x_1, ..., y_i, ..., x_k) \end{aligned}$$

and

$$M(x_1, x_2, ..., ax_i, ..., x_k) = aM(x_1, x_2, ..., x_i, ..., x_k).$$

A *0-linear operator* L_0, *on X*, is a constant function on X into X. That is, for some fixed $y \in X$, $L_0 x = y$ for all $x \in X$. We shall identify a 0-linear operator L_0 with its range so that $L_0 x = L_0$ for all $x \in X$.

Examples

1. Let $k(s, t_1, t_2, ..., t_n)$ be a square integrable function on the unit cube in E^{n+1} so that

$$\int_0^1 \int_0^1 \cdots \int_0^1 | k(s, t_1, ..., t_n)|^2 \, dt_1 \cdots dt_n \, ds < \infty.$$

Then

$$\begin{aligned} K(x_1, x_2, ..., x_n) \\ = \int_0^1 \int_0^1 \cdots \int_0^1 k(s, t_1, t_2, ..., t_n) \, x_n(t_n) \cdots x_2(t_2) \, x_1(t_1) \, dt_n \, dt_{n-1} \cdots dt_1, \end{aligned}$$

is an n-linear operator on $L^2[0, 1]$.

2. For $x \in Y = C^\infty[a, b]$, define

$$Dx = \frac{dx}{dt} + x.$$

Then the operator

$$K(x_1, x_2, \ldots, x_n) = \prod_{i=1}^{n} Dx_i$$

is an n-linear on Y.

3. Let C^n be the set of all n-tuples $(x_1, x_2, \ldots, x_n)$ of complex numbers. Let $(m_{i,j_1,j_2\ldots j_p})$, $i, j_1, \ldots, j_p = 1, 2, \ldots, n$, be a $(p+1)$-dimensional matrix of complex numbers. Given points $x^1, x^2, \ldots, x^p$ in C^n, where

$$\begin{aligned} x^1 &= (x_1{}^1, x_2{}^1, \ldots, x_n{}^1), \\ x^2 &= (x_1{}^2, x_2{}^2, \ldots, x_n{}^2), \\ &\vdots \\ x^P &= (x_1{}^P, x_2{}^P, \ldots, x_n{}^P), \end{aligned}$$

define

$$M(x^1, x^2, \ldots, x^P) = (a_1, a_2, \ldots, a_n),$$

where

$$a_i = \sum_{j_1, j_2, \ldots, j_p = 1} m_{i,j_1,j_2,\ldots,j_p} x_{j_p}^p x_{j_{p-1}}^{p-1} \cdots x_{j_1}^1 .$$

Then M is a p-linear operator on C^n.

Given a k-linear operator M on $X (k \geqslant 1)$ and an $x \in X$, we set

$$Mx^k = \underbrace{M(x, x, \ldots, x)}_{k \text{ times}}.$$

For each $k = 0, 1, \ldots, n$, let L_k be a k-linear operator on X. Then, the operator P on X into X, given by

$$Px = L_n x^n + L_{n-1} x^{n-1} + \cdots + L_1 x + L_0$$

is called an n-th *degree polynomial operator*.

3. Matrix Representations of Polynomials

Let $\ell^2(n)$ denote the set of square summable sequences of real (complex) numbers of length n (n may be infinity) with the usual inner product. The natural isometry ψ between $\ell^2(n)$ and a separable Hilbert space H of dimension n is a basis for the discussion of matrix representations of operators. If A is a continuous linear operator on H into H and ψ is the natural embedding of H into $\ell^2(n)$,then there is a naturally induced linear operator A' on $\ell^2(n,)$ given by

$$A'y = \psi A \psi^{-1}(y),$$

which is a *coordinate transformation*; the operator A takes the form of a matrix $(a_{ij})_{i,j=1}^n$. The correspondence between continuous operators A' and matrices (a_{ij}) is, however, only one-way when H is infinite dimensional. That is, not every matrix (a_{ij}) corresponds to a linear operator on $\ell^2(n)$. However, if $A' = (a_{ij})$ is what is known as a matrix representation of a linear operator A on H, then A' is a linear operator on ℓ^2 and both A and A' are continuous. See [1] and [4].

Completely analogous questions arise for k-linear operators and, hence, for polynomial operators (polynomials) on a separable Hilbert space. Since the proof of Theorem 5.5 requires only results on matrix representations of k-linear operators on finite dimensional Hilbert spaces (*unitary spaces*), we restrict ourselves to this case.

Let $\{\phi_j\}_{j=1}^n$ be an orthonormal basis for a unitary space H. Let M be a k-linear operator ($k \geqslant 1$) on H. Then M is said to have a *matrix representation* $(m_{j,i_1,i_2,\ldots,i_k})$ *with respect to* $\{\phi_j\}_{j=1}^n$ *if, for each k elements* $x^1, x^2, \ldots, x^k$ in H,

$$M(x^1, x^2, \ldots, x^k) = \sum_{j=1}^n d_j \phi_j ,$$

where

$$d_j = \sum \{m_{j,i_1,\ldots,i_k} x_{i_k}^k x_{i_{k-1}}^{k-1} \cdots x_{i_1}^1 : i_1, i_2, \ldots, i_k = 1, 2, \ldots, n\},$$

and

$$x^1 = \sum_{i=1}^n x_{i_1}^1 \phi_{i_1} ,$$

$$x^2 = \sum_{i=1}^n x_{i_2}^2 \phi_{i_2} ,$$

$$\vdots$$

$$x^k = \sum_{i=1}^n x_{i_k}^k \phi_{i_k} .$$

It is a simple matter to prove:

THEOREM 3.1. *Let H be a unitary space with an orthonormal basis* $\{\phi_i\}_{i=1}^n$. *Then*

(a) *All k-linear operators on H,* $k = 0, 1, 2,...,$ *are continuous.*

(b) *Each k-linear operator M, on H,* $k = 1, 2,...,$ *has a unique matrix representation* $(m_{j,i_1,...,i_k})$ *with respect to* $\{\phi_i\}_{i=1}^n$ *and*

$$m_{j,i_1,i_2,...,i_k} = (M(\phi_{i_k},..., \phi_{i_2}, \phi_{i_1}), \phi_j),$$

where (,) *denotes inner product.*

(c) *Every matrix* $(m_{j,i_1,i_2,...,i_k}), j, i_1,..., i_k = 1, 2,..., n,$ *is a matrix representation of a k-linear operator.*

The proof, being simple, is omitted.

At this point, we should remark that a 0-linear operator L_0, on H, can be assigned, too, a matrix representation. For if $y = \sum_{i=1}^n c_i\phi_i$ ($c_1, c_2,..., c_n$ being constants) is that fixed point of H for which

$$L_0 = L_0x = y$$

for all $x \in H$, we can associate with L_0 the "one dimensional matrix" (c_i). We shall call (c_i) a *matrix representation of* L_0. Conversely, with each n-tuple $(c_1, c_2,..., c_n)$ of constants we can associate a 0-linear operator L_0, on H, given by

$$L_0 = L_0x = \sum_{i=1}^n c_i\phi_i.$$

4. THE WEIERSTRASS THEOREM FOR REAL UNITARY SPACES

Let $\{\phi_j\}_1^n$ be an orthonormal basis for a real unitary space H and let F be a continuous function (operator) on H into H. Let $x \in H$, so that

$$x = \sum_{k=1}^n (x, \phi_k)\,\phi_k = \sum_{k=1}^n x_k\phi_k.$$

Then, since $F(x) \in H$, we have

$$F(x) = \sum_{i=1}^n f_i(x_1, x_2,..., x_n)\,\phi_i, \tag{1}$$

where each of the f_i is a continuous function on E^n.

Now, let P be a polynomial on H. Then, if $x = \sum_{i=1}^{n} x_i \phi_i$,

$$Px = \sum_{i=1}^{n} g_i(x_1, x_2, \ldots, x_n)\, \phi_i, \tag{2}$$

where each of the g_i's is a real polynomial in n variables. That is, if P is

of degree 0: $g_i(x_1, x_2, \ldots, x_n) = c_i$,

of degree 1: $g_i(x_1, x_2, \ldots, x_n) = \sum_{j=1}^{n} a_{ij} x_j + c_i$,

of degree 2: $g_i(x_1, x_2, \ldots, x_n) = \sum_{j,k=1}^{n} b_{ijk} x_k x_j + \sum_{j=1}^{n} a_{ij} x_j + c_i$

⋮

of degree m: $g_i(x_1, x_2, \ldots, x_n)$

$$= c_i + \sum_{k=1}^{m} \sum_{j_1, j_2, \ldots, j_k = 1}^{n} a_{i, j_1, j_2, \ldots, j_k} x_{j_k} \cdots x_{j_2} x_{j_1}.$$

This follows directly from Theorem 3.1 and from the definition of a matrix representation of a 0-linear operator. Namely, if

$$Px = L_0 + L_1 x + L_2 x^2 + \cdots + L_m x^m,$$

then

$$(a_{i, j_1, j_2, \ldots, j_k}),\; i, j_1, \ldots, j_k = 1, 2, \ldots, n,$$

is the matrix representation of $L_k (k \geqslant 1)$ and (c_i) is the matrix representation of L_0.

The polynomials $p(x_1, x_2, \ldots, x_n)$ in n real variables are dense in $C(\tilde{K})$, where $\tilde{K}$ is a compact subset of E^n. Using the natural embedding

$$\psi(x) = (x_1, x_2, \ldots, x_n)$$

of H into $l^2(n)$, we have that if K is compact in H, then $\psi(K) = \tilde{K}$ is compact in $l^2(n)$. For each $\epsilon > 0$ and for each i, there exists a polynomial

$$g_i(x_1, x_2, \ldots, x_n)$$

for which

$$\| f_i - g_i \| = \max_{x \in K} | f_i(x) - g_i(x) | < \epsilon / n^{1/2}. \tag{3}$$

Let N be a positive integer such that, for $i = 1, 2,..., n$, and for suitable constants c_i, $a_{i,j_1,j_2,...,j_p}$,

$$g_i(x_1, x_2, ..., x_n) = c_i + \sum_{p=1}^{N} \sum_{j_1,j_2,...,j_p=1}^{n} a_{i,j_1,j_2,...,j_p} x_{j_p} \cdots x_{j_2} x_{j_1} .$$

Invoking Theorem 3.1. (c), the matrix $(a_{i,j_1 \dots j_k})$ corresponds to a k-linear operator L_k on H. Define the polynomial P by

$$Px = L_0 + L_1 x + L_2 x^2 + \cdots + L_N x^N = \sum_{i=1}^{n} g_i(x_1, x_2, ..., x_n) \phi_i ,$$

where L_0 is the 0-linear operator $\sum_{i=1}^{n} c_i \phi_i$.
It then follows from (3) that

$$\| F(x) - Px \|_H^2 = \sum_{i=1}^{n} | f_i(x_1, ..., x_n) - g_i(x_1, ..., x_n)|^2$$

$$\leqslant \sum_{i=1}^{n} \| f_i - g_i \|^2 < \epsilon^2.$$

Thus,

$$\| F - P \| = \max_{x \in K} \| F(x) - Px \|_H < \epsilon,$$

and we have proved

THEOREM 4.1 (The Weierstrass Theorem for Unitary Spaces). *The polynomials on H, restricted to a compact subset K of the real unitary space H, are dense in the set C(K) of continuous functions on H into H restricted to K, where C(K) carries the uniform norm topology.*

5. EXTENTION OF THE WEIERSTRASS THEOREM TO A REAL, SEPARABLE HILBERT SPACE

Now let H be a real, infinite dimensional but separable Hilbert space with a complete, orthonormal basis $\{\phi_k\}_1^\infty$, and let P_n denote the *projection* of H onto $H_n = \text{span}\{\phi_1, \phi_2, ..., \phi_n\}$. That is

$$P_n x = \sum_{k=1}^{n} (x, \phi_k) \phi_k .$$

So P_n is a continuous, linear operator. Let F be a continuous function on H into H and define the function F_n, on H, by

$$F_n x = P_n F P_n x.$$

Clearly, F_n is continuous and of finite rank. This enables us to prove

LEMMA 5.1. *Let K be a compact set in H. Given $\epsilon > 0$, there exists a polynomial P on H, of finite rank, such that* $\max_{x \in k} \| F_n x - Px \| < \epsilon$.

Proof. Let $K_n = P_n(K)$. Then K_n is compact. Let $\tilde{F}_n$ be the restriction of F_n to H_n. Then $\tilde{F}_n$ is a continuous function on H_n into H_n and, as such, by Theorem 4.1, can be uniformly approximated by a polynomial $\tilde{P}$, of degree N, on H_n into H_n so that

$$\| \tilde{F}_n \tilde{x} - \tilde{P}\tilde{x} \| < \epsilon \tag{4}$$

for all $\tilde{x} \in K_n$. Here $\tilde{P}$ has the form

$$\tilde{P}\tilde{x} = \tilde{L}_0 + \tilde{L}_1 \tilde{x} + \cdots + \tilde{L}_N \tilde{x}^N.$$

Now extend P to all of H by defining

$$Px = \tilde{P} P_n x.$$

Clearly, P is of finite rank and

$$Px = L_0 + L_1 x + \cdots + L_N x^N$$

where $L_0 = \tilde{L}_0$ and $\tilde{L}_k \tilde{x}^k = L_k x^k (k = 1, 2, ..., N)$, $\tilde{x} = P_n x$. Thus P_n being linear, P is a polynomial on H. Thus

$$\begin{aligned} \| Px - F_n x \| &= \| \tilde{P} P_n x - P_n F P_n x \| \\ &= \| \tilde{P}\tilde{x} - P_n F P_n^2 x \| \\ &= \| \tilde{P}\tilde{x} - P_n F P_n \tilde{x} \| \\ &= \| \tilde{P}\tilde{x} - \tilde{F}_n \tilde{x} \| < \epsilon \end{aligned}$$

for all x in K.

To prove the Weierstrass Theorem for H, we must show that, given any compact set $K \subset H$ and $\epsilon > 0$, there exists a polynomial P on H such that $\| F - P \| < \epsilon$ in the uniform norm topology on $C(K)$. But

$$\| F - P \| \leqslant \| F - F_n \| + \| F_n - P \| .$$

By Lemma 5.1, we can make the second summand on the right less than $\epsilon/2$. We shall prove that the same is true for the first summand. We begin by recalling

THEOREM A (Dini's Theorem). *Let K be compact in H and let $\{f_n\}$ be a sequence of real-valued, continuous functions on K which converges to a continuous function f on K. Then, if $f_n(x) \geqslant f_{n+1}(x)$ for all $x \in K$ and for $n = 1, 2,\ldots$, then f_n converges uniformly to f on K.*

LEMMA 5.2. *Let K be compact in H. Then $P_n x$ converges uniformly to x on K.*

Proof of the Lemma. Let $f_n(x) = \| x - P_n x \|$. It is clear that the sequence $\{f_n\}$ satisfies the conditions of Theorem A.

LEMMA 5.3. *Let K be compact in H and let F be a continuous function on H into H. Then, the sequence $\{FP_n x\}$ converges uniformly to Fx on K.*

Proof. Let $\epsilon > 0$. Then, since F is continuous on H, for each x in K there exists a $\delta_x > 0$ such that

$$\| Fx - Fy \| < \frac{\epsilon}{2} \qquad \text{for all} \quad y \text{ in } N_{\delta_x}(x),$$

where $N_\delta(x) = \{y \in H : \| x - y \| < \delta\}$. The family $\{N_{\delta x/2}(x) : x \in K\}$ is an open cover of K. So there exists a finite subcover

$$\{N_{\delta(i)}(x_i) : i = 1, 2,\ldots, n\}, \qquad \text{where} \quad \delta(i) = \delta_{x_i}/2.$$

Let $\delta = \min\{\delta(i) : 1 \leqslant i \leqslant n\}$. Let $x \in K$ and $y \in H$ be so that $\| x - y \| < \delta$. Then, there exists some x_i such that $\| x - x_i \| < \delta(i)$.
But then

$$y - x_i \| \leqslant \| y - x \| + \| x - x_i \| < \delta(i) + \delta(i) = \delta_{x_i}$$

Thus,

$$\| Fx - Fx_i \| < \epsilon/2$$

and

$$\| Fy - Fx_i \| < \epsilon/2;$$

so that

$$\| Fx - Fy \| < \epsilon.$$

By the uniform convergence of $P_n x$ to x on K, there exists an n_0 such that, for all $x \in K$ and $n \geqslant n_0$,

$$\| x - P_n x \| < \delta.$$

But then, for all such x, n,

$$\| Fx - FP_n x \| < \epsilon.$$

LEMMA 5.4. *Let F be continuous on H and let K be compact in H. Then, given $\epsilon > 0$, there exists an integer $n_0 > 0$ such that*

$$\max_{x \in K} \| Fx - F_n x \| < \epsilon$$

whenever $n \geqslant n_0$.

Proof.

$$\begin{aligned} \| F_n x - Fx \| &= \| P_n F P_n x - Fx \| \\ &\leqslant \| P_n F P_n x - P_n F x \| + \| P_n F x - Fx \| \\ &\leqslant \| P_n \| \cdot \| F P_n x - Fx \| + \| P_n F x - Fx \| \\ &= \| F P_n x - Fx \| + \| P_n F x - Fx \| . \end{aligned}$$

Now apply Lemmas 5.2 and 5.3.

We next combine Lemmas 5.1 and 5.4. That is, let K be compact in H and let F be continuous on H. Given $\epsilon > 0$, there exists an integer $n > 0$ such that

$$\max_{x \in K} \| Fx - F_n x \| < \epsilon/2. \tag{5}$$

But by Lemma 5.1, there exists a continuous polynomial P such that

$$\max_{x \in K} \| F_n x - Px \| < \epsilon/2. \tag{6}$$

Combining (5) with (6), we have

$$\| Fx - Px \| < \epsilon,$$

for all $x \in K$. That is, we have proved

THEOREM 5.5. (*The Weierstrass Theorem for Real, Separable Hilbert Spaces*). *Let H be a real, separable Hilbert space. The family of continuous polynomials on H, restricted to a compact set K of H, is dense in the set $C(K)$ of continuous functions on H into H restricted to K, where $C(K)$ carries the uniform norm topology.*

6. Final Remarks

The classical Weierstrass Theorem has a complex analog. In fact, it can be shown that the family of polynomials in n complex variables and in their conjugates is dense in $C(K)$, where K is a compact subset of Z^n, Z denoting the complex plane. Consequently we conjecture that to secure a "Weierstrass Theorem" in complex Hilbert spaces would require the introduction of "conjugation operators." For example, if L_k is a k-linear operator ($k \geqslant 2$) then we would need to include among the polynomials operators, the operators

$$L_k^{n,p}x^k = L_k\bar{x}^n\bar{x}^p = L_k(\underbrace{\bar{x},\ldots,\bar{x}}_{n \text{ times}}\ \underbrace{x,\ldots,x}_{p \text{ times}}),$$

where $n + p = k$.

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